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The Structure of State Space With Respect to Imbedding

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The entanglement of formation as well as the conditional entropy can be used to define leaves in the state space, given by the linear superposition of their extremal points. Examples, where these leaves can be found and can be used to calculate the entanglement respectively the conditional entropy are presented. The definition of entanglement is generalized to infinite systems and allows again to find a leaf structure. Finally we remark on the additivity property of both expressions, offering a counter example to the additivity of the conditional entropy.

KEY WORDS: entanglement; conditional entropy; symmetry properties.

1. INTRODUCTION

The phenomenon of entanglement was already well known in the early stage of quantum mechanics (Schroedinger, 1936). In the near past it has again gained much interest of being a powerful resource in prospective quantum information techniques. There exist several expressions to quantify entanglement, depending what features should be described. One of them is entanglement of formation. It turns out that this expression not only serves to measure the costs to produce the entangled state (in the spirit of Bennet *et al.* (1996b) and Hayden *et al.* (2001) but also imposes a structure on the state space of the composite system, decomposing the state space into different leaves, a structure that we expect to be useful to evaluate strategies in quantum encoding (Benatti, 1996). Nevertheless not many examples have been studied so far. In this review we offer strategies to evaluate such leaves, we collect the known results and add a few additional ones. We also compare the structure of state space induced by entanglement with a similar one, induced by the conditional entropy. This structure is somehow opposite to the one induced by entanglement, but is not as rigged and especially does not satisfy additivity with respect to tensor products, a property that is one of the open questions in the theory of entanglement.

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2. ENTANGLEMENT OF FORMATION AND THE LEAF STRUCTURE OF STATE SPACE

We consider our quantum system to be described by an algebra of operators $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$. To avoid topological subtilities we assume in this chapter that the Hilbert space has finite dimensions. States over $\mathcal{M}$ are given by density matrices ρ such that $\omega(M) = Tr\rho M$. Entanglement of formation refers to a subalgebra $\mathcal{A} \subset \mathcal{M}$.

Definition 2.1. Given a subalgebra $\mathcal{A} \subset \mathcal{M}$. We define the entanglement of the state ω with respect to the subalgebra $\mathcal{A}$ by

$$E(\omega, \mathcal{M}, \mathcal{A}) = \inf \sum_i \lambda_i S(\omega_i) \mid \mathcal{A},$$

where the infimum is taken over all possible decompositions $\omega = \sum_i \lambda_i \omega_i$ of the state ω into states over $\mathcal{M}$.

Remark. (i) A special example corresponds to $\mathcal{M} = \mathcal{A} \otimes \mathcal{B}$, where usually the algebra $\mathcal{A}$ is assigned to Alice and $\mathcal{B}$ to Bob. In this situation

$$E(\omega, \mathcal{A} \otimes \mathcal{B}, \mathcal{A}) = E(\omega, \mathcal{A} \otimes \mathcal{B}, \mathcal{B}),$$

but we have also more general imbeddings $\mathcal{A} \subset \mathcal{M}$ in mind.

(ii) The entanglement of formation is a convex function of ω. With respect to $\mathcal{A}$ it is monotonically increasing, with respect to $\mathcal{M}$ it is monotonically decreasing (Narnhofer and Thirring, 1985).

(iii) Since the Hilbert space is finite the infimum is really achieved. The ω_i for which the infimum is achieved are called optimal decomposers.

The main observation that allows to impose a leaf structure on the state space and also enables us to evaluate the entanglement for a larger set of states is the following:

Theorem 2.1. *Let $f(\omega)$ be a concave function on ω. Let*

$$F(\omega) = \inf \sum_i \lambda_i f(\omega_i), \quad \sum_i \lambda_i \omega_i = \omega.$$

Then the state space S decomposes into leaves Ł, $S = \cup$ Ł and F is a linear functional on a leaf Ł, i.e.

$$F(\lambda\omega_1 + (1-\lambda)\omega_2) = \lambda F(\omega_1) + (1-\lambda)F(\omega_2), \quad \omega_1, \omega_2 \in \text{Ł}$$

Proof: From concavity it follows that the infimum is reached at extremal points, i.e., pure states. Linear decomposition of superpositions of states can only be better than the linear superposition of the decompositions, which makes F convex. Based on this observation different superpositions of one optimal decomposition can be compared and this gives the result (Benatti *et al.*, 1996a). Also we can remember that as a convex function F is the supremum over affine functionals, and these affine functionals can be labeled in our situation by Ł, $l_{Ł}(\omega) = F(\omega)$ for $\omega \in Ł$, (Benatti *et al.*, 2002).

We can collect some properties that these leaves have to satisfy.

(i) Let α_g, $g \in G$ be an automorphism group on $\mathcal{M}$ such that $\alpha_g \mathcal{A} \subset \mathcal{A}$ and $\omega \circ \alpha_g = \omega$. Let $\bar{\omega}$ belong to the leaf $Ł_\omega$. Then also $\bar{\omega} \circ \alpha_g \in Ł_\omega$.

(ii) We take pure states $\sigma_1, \ldots, \sigma_n$ and denote the corresponding vectors in the Hilbert space $\mathcal{H}$ by $|\sqrt{\sigma_1}\rangle, \ldots, |\sqrt{\sigma_n}\rangle$. □

Theorem 2.2. *Compatibility relation (Benatti and Narnhofer, 2001). We call states ω_1 and ω_2 compatible if they belong to the same leaf. $\sigma_1, \ldots, \sigma_n$ are extremal points of the same leaf if and only if*

$$\sum_i |\gamma_i|^2 S(|\sqrt{\sigma_i}\rangle\langle\sqrt{\sigma_i})|_{\mathcal{A}} + \sum_{ij}(\gamma_j \gamma_i^* |\sqrt{\sigma_i}\rangle\langle\sqrt{\sigma_j}| + \gamma_i \gamma_j^* |\sqrt{\sigma_j}\rangle\langle\sqrt{\sigma_i}|)|_{\mathcal{A}} \ln$$
$$\times |\sqrt{\sigma_i}\rangle\langle\sqrt{\sigma_i}|_{\mathcal{A}}) \leq \left\langle \sum_i \gamma_i \sqrt{\sigma_i} \right| \sum_j \gamma_j \sqrt{\sigma_j} \left. \vphantom{\sum_i} \right\rangle$$
$$\times S\left(\frac{|\sum_i \gamma_i \sqrt{\sigma_i}\rangle\langle\sum_j \gamma_j \sqrt{\sigma_j}}{\langle\sum_i \gamma_i \sqrt{\sigma_i}| \sum_j \gamma_j \sqrt{\sigma_j}\rangle} \right|\right)$$

for all possible $\gamma_i \in \mathcal{C}$. □

The proof can be found in (Benatti and Narnhofer, 2001). It is based on perturbation around the optimal decomposition together with an application of Theorem 2.1.

As a special case we consider the values $\gamma_1 = 1$, $\gamma_2 = \epsilon$ all other $\gamma_i = 0$. Then we can expand the inequality. Up to order ε it reduces to the equality

$$Tr|\sqrt{\sigma_1}\rangle\langle\sqrt{\sigma_2}|(\ln|\sqrt{\sigma_1}\rangle\langle\sqrt{\sigma_1}| - \ln|\sqrt{\sigma_2}\rangle\langle\sqrt{\sigma_2}|) = 0.$$

Up to second order in ε an inequality remains, that is not much more transparent than the general inequality. It is an open problem whether the above inequality cannot be reduced to a smaller set of γ_i, e.g., if the compatibility of all pairs of pure states ($\gamma_i = 0$ for all but two elements) guarantees already that the pure states generate a leaf. So far no counterexample is known, and in the next chapter we will offer an example where the leaf is really found on the basis of this assumption.

3. FINITE DIMENSIONAL EXAMPLES

(A) The simplest example is provided by $\mathcal{M} = M_n \otimes M_k^0$, where M_n is a n dimensional full matrix algebra and M_k^0 an abelian algebra of dimension k. Then any state ω can be decomposed into $\omega = \sum_{l=1}^{k} \lambda_l \omega_l$ with ω_l a pure state on $M_k^0 \cdot \omega_l$ can further be decomposed into pure states over M_n. Therefore

$$E(\omega \mathcal{M}, M_n) = E(\omega, \mathcal{M}, M_k^0) = 0$$

and the state space consists only of one leaf, all pure states being compatible.

(B) We take $\mathcal{M} = M_2$ and $\mathcal{A} = M_2^0 = \{\sigma_z\}$ with the notation of Pauli matrices. Every state over M_2 corresponds to a density matrix in M_2. We choose the special states ω with $\omega \circ \alpha = \omega$, where α is the automorphism $\alpha\sigma_z = -\sigma_z, \alpha\sigma_x = \sigma_x$. If $|z_1, z_2\rangle$ is an optimal decomposer so is $|z_2, z_1\rangle$. To every above ω we can find an appropriate pair of such states and can convince us that this pair satisfies the necessary compatibility relation (Benatti *et al.*, 1996a). Therefore the corresponding leaf consists of the orbit under α of one state and further more the whole state space can be covered by these leaves after rotation in the *xy* space.

This example provides us with a possible strategy to search for optimal decompositions, though it is only applicable if we want to decompose a state with good symmetry properties.

Assume $\omega \circ \alpha_g = \omega \forall_g \in G$. We look for a pure state $\bar{\omega}$ such that $\omega = \int d\eta_g \bar{\omega} \circ \alpha_g$, i.e., we look for a state whose orbit under the symmetry group generates the leaf. If the group is large the orbit might be large too, therefore the compatibility condition (Theorem 2.2) might be too demanding on the many $\bar{\omega} \circ \alpha_g$. Therefore we look for a subgroup $H \subset G$ such that $\bar{\omega} \circ \alpha_h = \bar{\omega} \forall h \in H$. Therefore the orbit reduces to $G \mid H$ and should be small enough to satisfy all compatibility relations but large enough to generate ω. In addition we have to be aware that the leaf might be generated by several orbits. That this strategy can be successful but that all possibilities we mentioned can be realized will be demonstrated in the following example:

(C) We take $\mathcal{M} = M_3$ and $\mathcal{A} = M_3^0$. The relevant group is the permutation group which is of order 6. The states $\omega \circ \alpha_\pi = \omega$ are labeled by one parameter, $\omega(e_{ii}) = 1/3, \omega(e_{ij}) = z$ for $i \neq j$ and $-1/6 \leq z \leq 1/3$ so that the state is positive. For an optimal decomposition we need at least three states, at most nine. If three states are sufficient, the pure state has to be invariant under a subgroup $H \subset G$, e.g., without loss of generality we take the permutations (2, 3). This fixes the possible pure state uniquely depending on z. But it turns out (Benatti *et al.*, 1996b, 1999) that this decomposition is not always optimal. We have two bifurcation points (Benatti *et al.*, 2002; Terhal and Vollbrecht, 2000) $-1/6 < z_0 < 0 < z_1 < 1/3$. This is a result of numerical analysis but can be made plausible by the following observation:

For $z = 0$ the tracial state can be decomposed into eigenvectors of M_3^0, thus one state corresponds to (1, 0, 0) and gives entanglement $E(\omega_0) = 0$. For $z = 1/3$ the state is already pure and the corresponding vector is $1/\sqrt{3}(1, 1, 1) = \psi_{1/3}$. For $z = -1/6$ it is easy to find that $\psi_{-1/6} = 1/\sqrt{2}(1, -1, 0)$ is an optimal decomposer.

Here the state is invariant under the group H but not the vector, only its ray. It is possible to pass continuously from $\psi_{1/3}$ to ψ_0 remaining a fix point of H but not from ψ_0 to $\psi_{-1/6}$. This explains that a bifurcation value has to occur. The accurate value varies if we vary the concave function $f(\omega)$ in Theorem 2.1 and can therefore not be explained by general arguments. The bifurcation point z_1 is of different nature. Here we do not break the symmetry of H but we start to need two orbits with varying weight. This bifurcation point can be found by a mapping $\Gamma : M_2 \to M_3; M_2^0 \to M_3^0$

$$\begin{pmatrix} a & c \\ c & d \end{pmatrix} \to \begin{pmatrix} a & c/2 & c/2 \\ c/2 & b/2 & b/2 \\ c/2 & b/2 & b/2 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} \to \begin{pmatrix} a \\ b/2 \\ b/2 \end{pmatrix}.$$

Every decomposition of $\Gamma(\rho)$ is again into density matrices of the above form and satisfies especially that $S(\Gamma(\omega_i)) \mid_{M_3^0}$ has the same monotonicity behavior with respect to the relevant parameters as $S(\omega_i) \mid_{M_2^0}$. Therefore an optimal decomposition over M_2 can be mapped into an optimal decomposition over M_3. Especially

$$\Gamma 1/\sqrt{3}(1, \sqrt{2}) = 1/\sqrt{3}(1, 1, 1), \quad \Gamma 1/\sqrt{3}(\sqrt{2}, 1) = 1/\sqrt{6}(2, 1, 1).$$

These two vectors combine in three-dimensional case to a leaf, but they also belong to the leaf that is defined by $\omega_{1/3}$ respectively to the leaf defined by ω_{z_0}. Their orbits under the permutation group generate the leaf for all ω_z, $z_0 \leq z \leq 1/3$ which can be checked by comparing with a decomposition of just one orbit for $z_0 < z < 1/3$. This example is in support to the conjecture that a leaf is determined by the pairs of its extremal points. (Compare the remark after Theorem 2.2.)

4. INFINITE ALGEBRAS

Though in quantum information theory normally one restricts oneself to finite dimensional algebras it seems worthwhile to examine how increasing dimensions might influence the structure and especially whether similar considerations also give some insight when infinite algebras are imbedded in one another. In this situation the first problem arises in the definition of the entanglement, qualitatively and quantitatively, because pure states on infinite von Neumann algebras do not exist.

(A) Let us first consider a simple imbedding: let $\mathcal{A}$ be a type II_1 factor algebra and α a free automorphism (therefore not an inner automorphism) with $\alpha^2 = 1$ and $\mathcal{A}$ imbedded into the algebra $\mathcal{M} = \mathcal{A} \rtimes_\alpha Z^2$, i.e., the crossed product of the algebra $\mathcal{A}$ with the automorphism α and by the assumptions again a type II_1 factor. A physical realization is given with $\mathcal{M}$ the algebra of infinitely many fermions and $\mathcal{A}$ the subalgebra of even polynomials in creation and annihilation operators where α is induced by some $(a_0 + a_0^*)$. We can write elements of $\mathcal{M}$ respectively of $\mathcal{A}$ conveniently as

$$M = \begin{pmatrix} A_1 & A_2 \\ \alpha A_2 & \alpha A_1 \end{pmatrix},$$

where A_1, A_2 belongs to $\mathcal{A}$, and A is imbedded into $\mathcal{M}$ by demanding that $A_2 = 0$. On $\mathcal{M}$ we can define an automorphism $\hat{\alpha}$

$$\hat{\alpha} M = \hat{\alpha} \begin{pmatrix} A_1 & A_2 \\ \alpha A_2 & \alpha A_1 \end{pmatrix} = \begin{pmatrix} A_1 & -A_2 \\ -\alpha A_2 & \alpha A_1 \end{pmatrix},$$

so that $\mathcal{A}$ is the fix point algebra under $\hat{\alpha}$. Notice that the automorphism α can now be implemented by either of the operators

$$\begin{pmatrix} V & 0 \\ 0 & V \end{pmatrix} \text{or} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where for the first operator $V \notin \mathcal{A}$ but the later operator belongs to $\mathcal{M}$.

To find a definition for the entanglement let us recall the definitions in the finite case: the entropy itself can be written (Narnhofer and Thirring, 1985) as

$$S(\omega) = \sup \sum_k \lambda_k S(\omega \mid \omega_k), \quad \omega = \sum_k \lambda_k \omega_k,$$

where the supremum is taken over all possible decomposition and is reached for every decomposition into pure states. The entanglement then reads

$$E(\omega, \mathcal{M}, \mathcal{A}) = \inf \sum_i^{\mathcal{M}} \mu_i \sup \sum_k^{\mathcal{A}} \lambda_{ki} S(\omega_i \mid \omega_{ik}).$$

Here $\omega = \sum \mu_i \omega_i$ is decomposed into states over $\mathcal{M}$ whereas $\omega_i = \sum \lambda_{ik} \omega_{ik}$ is decomposed into states over $\mathcal{A}$. Every decomposition results from a positive operator in the relative commutant of a representation in which the state is given as expectation value with a vector

$$\omega(A) = \langle \Omega | \Pi(A) | \Omega \rangle \quad \omega_k(A) = \langle \Omega | Q_k \Pi(A) | \Omega \rangle,$$

where $Q_k \in \Pi(A)'$, $Q_k \geq 0$. We can now replace the definition of the entanglement by

$$E(\omega, \mathcal{M}, \mathcal{A}) = \inf \sum_i^{\mathcal{M}} \mu_i \sup \sum_k^{\mathcal{A}} \inf_{E_i} \omega_i(Q_k) S(\omega_i(E_i(Q_k)\cdot) | \omega_i(Q_k \cdot))|_{\mathcal{A}},$$

where we stay in a common representation for all ω_i. Here $E_i(Q_k)$ is an ω_i preserving completely positive map from $\Pi(A)'$ into $\Pi(\mathcal{M})'$ and the supremum is taken over all decompositions $\sum_k Q_k = 1$ of operators $Q_k > 0 \in (\mathcal{A})'$. Therefore Q_k contributes to the entanglement only as far as it is a refinement of a decomposition into states over $\mathcal{M}$. Since the infimum is still achieved if ω_i is pure over $\mathcal{M}$ (if $\mathcal{M}$ is finite dimensional so that this statement makes sense) and then $\omega_i(E_i(Q_k)\cdot) = \omega_i(Q_k)\omega_i(\cdot)$ the two definitions coincide in the finite dimensional case. Especially also in this form Theorem 2.1 can be applied. But in the infinite case it enables us to stop with a decomposition into ω_i already at an early stage as we will see in the following examples. First we note

Lemma 4.3. *For the algebras $\mathcal{M} = \mathcal{A} \bowtie_\alpha Z^2 \supset \mathcal{A}$ the entanglement of any state $\hat{\omega}$ satisfies $E(\hat{\Omega}, \mathcal{M}, \mathcal{A}) \leq \ln 2$.*

Proof: Let A' be an operator in the relative commutant $\pi(\mathcal{A})'$ in the GNS representation induced by the state ω over $\mathcal{A}$, where we assume that ω is faithful, i.e., $\omega(\mathcal{A}) > 0$ for all positive operators $A \in \mathcal{A}$. Then for any extension $\hat{\omega}$ of ω as state over $\mathcal{M}$ we can write the elements of $\Pi(M)'$ respectively of $\Pi(\mathcal{A})' \supset \mathrm{II}(\mathcal{M})'$ as

$$\begin{pmatrix} A_1' & A_2'V \\ A_2'V & A_1' \end{pmatrix} \in \Pi(\mathcal{M})' \begin{pmatrix} A_1' & A_2'V \\ A_3' & \alpha A_4' \end{pmatrix} \in \Pi(\mathcal{A})',$$

where the automorphism α is implemented by $\alpha A = VAV$. On $\Pi(\mathcal{A})'$ there exists the automorphism

$$\overline{\alpha}\begin{pmatrix} A_1' & A_2'V \\ A_3'V & A_4' \end{pmatrix} = \begin{pmatrix} A_4' & A_3'V \\ A_2'V & A_1' \end{pmatrix},$$

such that $E(A)' = \frac{1+\overline{\alpha}}{2}A' \in \Pi(\mathcal{M})'$ is a conditional expectation from $(\Pi(A)'$ into $\Pi(\mathcal{M})'$ that satisfies $E(Q) \geq \frac{1}{2}Q$. Since every state over $\mathcal{M}\hat{\omega}$ can be written in the form

$$\left\langle \begin{matrix} \Omega| & \cdots & |\Omega \\ \Psi| & \cdots & |\Psi \end{matrix} \right\rangle + \left\langle \begin{matrix} V\Psi| & \cdots & V|\Psi \\ V\Omega| & \cdots & V|\Omega \end{matrix} \right\rangle.$$

It follows that with

$$\left\langle \begin{matrix} 0 & V \\ V & 0 \end{matrix} \right\rangle \left| \begin{matrix} V\Psi \\ V\Omega \end{matrix} \right\rangle = \left| \begin{matrix} \Omega \\ \Psi \end{matrix} \right\rangle$$

the state $\hat{\omega}$ corresponds to a state over $\Pi(\mathcal{A})'$ for which $\hat{\omega} \circ \overline{\alpha} = \hat{\omega}$ and therefore $\hat{\omega}(E(Q)) = \hat{\omega}(Q)$. Together with the general estimate on the relative entropy that $S(\omega \mid \phi) < 0$ if $w > \phi$ this proves the lemma. We want to calculate the entanglement for special states and to find the corresponding leaf.

(a) Let $\hat{\omega}(M) = \langle\hat{\Omega}|M|\hat{\Omega}\rangle$ satisfy $\hat{\omega} \circ \hat{\alpha} = \hat{\omega}$, i.e. we consider gauge invariant states over $\mathcal{M}$. All these states belong to the same leaf and satisfy $E(\hat{\omega}, \mathcal{M}, \mathcal{A}) = 0$. □

Proof: The set of these states is stable under linear superposition. Further with

$$\hat{\omega}(A) = \left\langle \begin{matrix} \Omega \\ 0 \end{matrix} \middle| \begin{pmatrix} A_1 & 0 \\ 0 & \alpha A_1 \end{pmatrix} \middle| \begin{matrix} \Omega \\ 0 \end{matrix} \right\rangle, \; E(Q) = E \begin{pmatrix} A_1' & A_2' V \\ A_3' V & A_4' \end{pmatrix} = \begin{pmatrix} A_1' & 0 \\ 0 & A_1' \end{pmatrix},$$

$$\hat{\omega}(E(Q)A) = \hat{\omega}(QA),$$

so that decompositions by projectors from $\Pi(\mathcal{A})'$ reduce to decompositions already in $\mathcal{M}$.

(b) Consider states of the form

$$\hat{w}(M) = \left\langle \begin{matrix} \Psi| & \cdots & |\Psi \\ \Psi| & \cdots & |\Psi \end{matrix} \right\rangle.$$

All states of this form belong to the same leaf and for them $E(\hat{\omega}, M, \mathcal{A}) = \ln 2$. □

Proof: Every vector that is dominated by $\hat{\omega}$ can be represented by a vector obtained by the application of some vector from $\Pi = \mathcal{M}'$, $\tilde{\omega}(M) = \tilde{\omega}(M^{'*} M M')$ where therefore $\tilde{\omega}$ is now implemented by the vector

$$\begin{pmatrix} A_1' & A_2' V \\ A_2' V & A_1' \end{pmatrix} \left| \begin{matrix} \Psi \\ \Psi \end{matrix} \right\rangle = \left| \begin{matrix} (A_1' + A_2' V)\Psi \\ (A_1' + A_2' V)\Psi \end{matrix} \right\rangle$$

and is therefore of the desired form. The lemma follows if for all these states we can find an appropriate decomposition such that for all E

$$\sum_k \hat{\omega}(Q_k) S(\hat{\omega}(E(Q_k)\cdot \mid \hat{\omega}(Q_k \cdot) = \ln 2.$$

Let us assume that $|\Psi\rangle = C'|\Omega\rangle$ for some $C' \in \mathcal{A}'$ and $|\Omega\rangle$ is the vector implementing the tracial state on $\mathcal{A}$. Take a projection in $\mathcal{A}'$, P' with $\alpha P' = 1 - P'$ and $[P', C'] = 0$. Such a projection can be found fore a dense set of C'. Then for any E

$$E \begin{pmatrix} P' & 0 \\ 0 & 1 - P' \end{pmatrix} = \begin{pmatrix} \overline{P'} & 0 \\ 0 & \overline{P'} \end{pmatrix}$$

for some projector $\overline{P}'$.

$$\left\langle \begin{matrix} \Omega \\ \Omega \end{matrix} \right| \begin{pmatrix} C'^* & 0 \\ 0 & C'^* \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & \alpha A_1 \end{pmatrix} \begin{pmatrix} P' & 0 \\ 0 & 1 - P' \end{pmatrix} \begin{pmatrix} C' & 0 \\ 0 & C' \end{pmatrix} \left| \begin{matrix} \Omega \\ \Omega \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \Omega \\ \Omega \end{matrix} \middle| \begin{pmatrix} C'^* C' & 0 \\ 0 & C'^* C' \end{pmatrix} \middle| \begin{matrix} \Omega \\ \Omega \end{matrix} \right\rangle \langle \omega | A_1 (P' + \alpha(1 - P')|\omega\rangle$$

$$= c\langle \Omega | A_1 P' | \Omega \rangle = \omega_1(A_1)$$

for appropriately chosen operators A_1, P' that cluster with C' whereas $\hat{\omega}(E(Q)A)$ as state over $\mathcal{A}$ is α invariant. Based on the Kosaki formula for the relative entropy with appropriate variation on A_1, P' the decomposition by P' is as powerful as the decomposition of the tracial state into a pure state for a two dimensional matrix algebra and we can achieve the maximal value $\ln 2$.

(c) We consider the state $\hat{\omega}_U$ induced by the vectors $|\begin{smallmatrix} U\Psi \\ \alpha U\Psi \end{smallmatrix}\rangle$ with $\alpha U \neq U$. These states belong to a leaf $Ł_U$ on which again $E(\hat{\omega}_U, \mathcal{M}, \mathcal{A}) = \ln 2$. The leaves $Ł_U \neq Ł_1$. □

Proof: The leaf $Ł_U$ results from the automorphism γ_U implemented by $(\begin{smallmatrix} U & 0 \\ 0 & \alpha U \end{smallmatrix})$ that satisfies $\gamma_u \mathcal{A} \subset \mathcal{A}$ and therefore also acts as map between leaves. The leaves have to be different because a linear superposition of two states of different leaves dominates a state with vanishing entanglement

$$\hat{\omega}_U + \hat{\omega}_1 \geq c_U \hat{\omega}(1 + \hat{\alpha}).$$

Consider the states in the leaves that are obtained from the tracial state by operators from $\prod(\mathcal{M})'$, $(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) + (\begin{smallmatrix} 1 & U'VU'^* \\ U'VU'^* & 1 \end{smallmatrix})$. In the spectral representation taking into account that $U'VU'^* \neq 1$ is selfadjoint and unitary $(\begin{smallmatrix} 2 & 1\pm 1 \\ 1\pm & 2 \end{smallmatrix})$ we see that in some subspace it acts as the identity and cannot break the invariance of the initial state under $\hat{\alpha}$.

This does not implement that $Ł_U$ and $Ł_1$ have trivial intersection, e.g. we can imagine there exists a Ψ such that Ψ and $U\Psi$ are orthogonal for all U.

Collecting the results for the imbedding $\mathcal{A} \subset \mathcal{M} = \mathcal{A} \rtimes_\alpha Z^2$ we notice that the amount of entanglement varies as for $M_2^0 \subset M_2$. But to every value of entanglement there belong infinitely many different leaves reflecting the size of the algebra.

(B) As a completely different example we can consider the imbedding $\mathcal{A} \subset \mathcal{M} = \mathcal{A} \otimes \mathcal{B}$ where both algebras $\mathcal{A}$ and $\mathcal{B}$ are infinite algebras. Here we have not succeeded to find a closed expression for the entanglement. We can only define

$$E(\omega, \mathcal{A} \otimes \mathcal{B}, \mathcal{A}) = \sup_n E(\omega, \mathcal{A}_n \otimes \mathcal{B}_n, \mathcal{A}_n),$$

where $\mathcal{A}_n$ and $\mathcal{B}_n$ are finite dimensional subalgebras (Narnhofer, 2002). The supremum can be replaced by taking the limit over any sequence of increasing algebras as a consequence of the monotonicity properties of the entanglement. (Compare Narnhofer (2002) with a more detailed analysis.) □

5. THE CONDITIONAL ENTROPY

Another quantitiy that behaves differently in quantum theory than in classical theory is the conditional entropy. In classical theory it is is defined by

$$H_\omega(\mathcal{M} \mid \mathcal{A}) = S(\omega)|_{\mathcal{M}} - S(\omega)|_{\mathcal{A}},$$

which can be generalized to

$$H_\omega(\mathcal{B} \mid \mathcal{A}) = S(\omega)|_{\mathcal{B}\vee\mathcal{A}} - S(\omega)|_{\mathcal{A}}$$

if we do not consider imbeddings. This expression does not work in quantum theory, on one hand by lack of monotonicity of the entropy, on the other hand because the algebra $\mathcal{B} \vee \mathcal{A}$ generated by the two subalgebras will in general be too big. As a useful replacement one considers (Ohya and Petz, 1993)

$$H_\omega(\mathcal{B} \mid \mathcal{A}) = \sup \sum_i \lambda_i [S(\omega \mid \omega_i)|_{\mathcal{B}} - S(\omega \mid \omega_i)|_{\mathcal{A}}],$$

where the supremum is taken over all possible decompositions $\omega = \sum_i \lambda_i \omega_i$ into states over $\mathcal{M}$ or $\mathcal{A} \vee \mathcal{B}$. Different from classical theory we can find states for which

$$H_\omega(\mathcal{A} \otimes \mathcal{B} \mid \mathcal{A}) > H_\omega(\mathcal{B} \mid \mathcal{A}).$$

The optimal decomposition for $H_\omega(\mathcal{B}|\mathcal{A})$ asks for a delicate balance not to be too fine for $\mathcal{A}$ but sufficiently fine for $\mathcal{B}$. If however we concentrate on imbeddings $\mathcal{A} \subset \mathcal{M}$ then $H_\omega(\mathcal{M} \mid \mathcal{A})$ has some analogies with the entanglement.

With

$$H_\omega(\mathcal{M} \mid \mathcal{A}) = \sup \sum_i \lambda_i [S(\omega \mid \omega_i)|_{\mathcal{M}} - S(\omega \mid \omega_i)|_{\mathcal{A}}],$$

the conditional entropy is concave in ω and the supremum is achieved for pure states ω_i. This can be seen by the following observations: Refinement of the decomposition improves the estimate because

$$\sum_i \sum_j \lambda_{ij} S(\omega \mid \omega_{ij}) = \sum_i \sum_j \lambda_{ij} S(\omega \mid \omega_i) + \sum_i \sum_j \lambda_{ij} S(\omega_i \mid \omega_{ij})$$

and

$$S(\omega_i|\omega_{ij})|_{\mathcal{M}} - S(\omega_i|\omega_{ij})|_{\mathcal{A}} \geq 0$$

for $\mathcal{M} \supset \mathcal{A}$. For pure states ω_i

$$\sum_i \lambda_i [S(\omega \mid \omega_i)|_{\mathcal{M}} - S(\omega \mid \omega_i)|_{\mathcal{A}})] = S(\omega)|_{\mathcal{M}} - S(\omega)|_{\mathcal{A}} + \sum \lambda_i S(\omega_i)|_{\mathcal{A}}.$$

For the last expression we have to look for the supremum instead of looking for the infimum as we did for calculating the entanglement. We can apply a variational principle (B. Kuemmerer and R. Werner, personal communication, 1995) that is conclusive as long as we do not reach the boundary of the area of permitted decompositions. This boundary will not be reached if we limit the number of states in the decomposition sufficiently. The variation of the entropy defines a vector valued function (compare also Benatti *et al.* (1996a)

$$F(|\phi\rangle) = \frac{\partial}{\partial < \psi|} \|\psi\|^2 S\left(\frac{|\phi+\psi\rangle\langle\phi+\psi|}{\langle\phi+\psi|\phi+\psi\rangle}\right)$$

that satisfies $F(c|\phi\rangle) = cF(|\phi\rangle)$. Together with the condition $\rho = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$ this reduces to a kind of eigenvalue equation

$$F(|\phi_i\rangle) + M(\rho)|\phi_i\rangle = 0$$

with $M(\rho)$ acting as Lagrange multiplier. Because of linearity it follows that with $\sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$ being an optimal decomposition also $\sum_i \mu_i |\phi_i\rangle\langle\phi_i|$ is an extremal decomposition for some $\overline{\rho}$, i.e., $M(\rho) = M(\overline{\rho})$ serves as Lagrange multiplier also for the new $\overline{\rho}$. Of course we have to keep the possibility in mind that a supremum might change into a saddle point. Apart from this restriction we can conclude that if a set (ω_i) is optimal with respect to $\omega = \sum_i \lambda_i \omega_i$ then it is also optimal with respect to $\overline{\omega} = \sum_i \mu_i \omega_i$. In this situation the conditional entropy also defines leaves in the state space.

If we look for a compatibility condition similar as for the entanglement then it just turns into the opposite inequality

$$\begin{aligned}\sum_i |\gamma_i|^2 S(|\sqrt{\sigma_i}\rangle\langle\sqrt{\sigma_i}|_{\mathcal{A}}) + \sum_{i,j} (\gamma_i\gamma_j^* |\sqrt{\sigma_i}\rangle\langle\sqrt{\sigma_j}|) + (\gamma_j\gamma_i^* |\sqrt{\sigma_j}\rangle\langle\sqrt{\sigma_i}|)_{\mathcal{A}} \\ \ln \times |\sqrt{\sigma_i}\rangle\langle\sqrt{\sigma_i}|_{\mathcal{A}}) \geq \left\langle \sum_i \gamma_i \sqrt{\sigma_i} | \sum_j \gamma_j \sqrt{\sigma_j} \right\rangle \\ \times S\left(\frac{|\sum_i \gamma_i \sqrt{\sigma_i}\rangle\langle\sum_j \gamma_j \sqrt{\sigma_j}}{\langle\sum_i \gamma_i \sqrt{\sigma_i}| \sum_j \gamma_j \sqrt{\sigma_j}}\right)\end{aligned}$$

now with the restriction that may be the inequality only holds for a restricted area of γ_i.

The similarity of the compatibility relation is of interest in the context of one of the open problems in the theory of entanglement: is the entanglement additive, i.e., is

$$E(\omega_1 \otimes \omega_2, \mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{A}_1 \otimes \mathcal{A}_2) = E(\omega_1, \mathcal{M}_1, \mathcal{A}_1) + E(\omega_2, \mathcal{M}_2, \mathcal{A}_2)?$$

Known examples support the conjecture. Also if $E(\omega_2, \mathcal{M}_2, \mathcal{A}_2) = 0$ then equality follows from

$$\begin{aligned} &E(\omega_1 \otimes \omega_2, \mathcal{A}_1 \otimes \mathcal{B}_1 \otimes \mathcal{A}_2 \otimes \mathcal{B}_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \\ &\quad\geq E(\omega_1 \otimes \omega_2, \mathcal{A}_1 \otimes \mathcal{B}_1 \otimes \mathcal{A}_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \\ &\quad= E(\omega_1 \otimes \omega_2, \mathcal{A}_1 \otimes \mathcal{B}_1 \otimes \mathcal{A}_2, \mathcal{B}_1)\end{aligned}$$

$$\geq E(\omega_1 \otimes \omega_2, \mathcal{A}_1 \otimes \mathcal{B}_1, \mathcal{B}_1)$$
$$= E(\omega_1 \otimes \omega_2, \mathcal{A}_1 \otimes \mathcal{B}_1, \mathcal{A}_1).$$

In a more general situation the additivity of entanglement translated to the leaf structure to tensor products would demand that with (σ_1, σ_2) belonging to a leaf of one part and (ρ_1, ρ_2) belonging to a leaf of the other then $\sigma_1 \otimes \rho_1$ and $\sigma_2 \otimes \rho_2$ have to belong to the same leaf in the tensor product. In the inequality $S(|\sum \gamma_i \sqrt{\sigma_i} \otimes \sqrt{\rho_i}\rangle)$ is the only term that does not factorize and has to be estimated on the basis of $S(|\sum_i \gamma_i' \sqrt{\sigma_i}\rangle)$ and $S(|\sum_i \gamma_i'' \sqrt{\rho_i}\rangle)$. Such an estimate is missing so far. But it can support additivity either for the entanglement or for the conditional entropy. But for the conditional entropy we will give already a counter example to additivity. This example shows that provided some relation between the entropies above exist then it can only support additivity of the entanglement.

Example. Consider the tracial state on $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$ with $\mathcal{A} = M_{n^2}, \mathcal{B} = M_n$, $\mathcal{C} = M_n$.

Then

$$H_\tau(\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \mid \mathcal{B} \otimes \mathcal{C}) = 4 \ln n, \quad H_\tau(\mathcal{A} \otimes \mathcal{B} \mid \mathcal{B}) = 2 \ln n.$$

whereas with $H_\tau(\mathcal{C} \mid \mathcal{C}) = 0$ additivity would demand identity of the two expressions.

At last we present a simple example where the conditional entropy can be calculated on the basis of similar considerations as for the entanglement and really gives a leaf structure in the state space that is in some sense opposite to the one defined by the entanglement.

Example. Consider $M_n \supset M_n^0 = [P_i, i = 1, \ldots, n]$. Take $\rho = \lambda_i P_i$. This state is invariant under unitary transformations $U \in M_n^0$. Therefore we can take $M_n^0 = G$, the group under consideration that generates the orbit in the leaf. Take Q a one dimensional projector that satisfies $Tr\, Q P_i = 1/n$, e.g., $Q = 1/n|1, \ldots, 1\rangle \langle 1, \ldots, 1|$. The orbit of Q defines a complete set of vectors in the Hilbert space and we can pick $Q_1, \ldots, Q_n$ with $\sum_i Q_i = 1$. Therefore $\sqrt{\rho} Q_i \sqrt{\rho}$ decomposes ρ and satisfies

$$S\left(\frac{\sqrt{\rho} Q_i \sqrt{\rho}}{Tr \rho Q_i}\right) = S(\rho).$$

Taking into account the concavity of the entropy we have therefore achieved the optimal decomposition and

$$H_\omega(M_n \mid M_n^0) = S(\omega) |_{M_n}.$$

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